

RAINBOW RAMSEY THEOREM FOR TRIPLES IS STRICTLY WEAKER THAN THE ARITHMETICAL COMPREHENSION AXIOM

WEI WANG

ABSTRACT. We prove that $\text{RCA}_0 + \text{RRT}_2^3 \not\vdash \text{ACA}_0$ where RRT_2^3 is the Rainbow Ramsey Theorem for 2-bounded colorings of triples. This reverse mathematical result is based on a cone avoidance theorem, that every 2-bounded coloring of pairs admits a cone-avoiding infinite rainbow, regardless of the complexity of the given coloring. We also apply the proof of the cone avoidance theorem to the question whether $\text{RCA}_0 + \text{RRT}_2^4 \vdash \text{ACA}_0$ and obtain some partial answer.

1. INTRODUCTION

For computability theorists, Ramsey's theorem has been attractive for decades since Specker's work [12]. In this pioneering work, Specker showed that a computable 2-coloring of pairs may admit no computable infinite homogeneous set. Let us recall some concepts from Ramsey theory. We use $[X]^n$ to denote the set of n -element subsets of X where $n \leq \omega$; a function $f : [\omega]^n \rightarrow k$ is also called a k -coloring, and a set X is f -homogeneous if f is constant on $[X]^n$.

Ramsey's Theorem. *If $n, k \in \omega$ and f is a k -coloring of $[\omega]^n$, then there exists an infinite f -homogeneous set.*

The instance of Ramsey's theorem for specific n and k is denoted by RT_k^n . As a consequence of Specker's work, $\text{RCA}_0 \not\vdash \text{RT}_2^2$. Here RCA_0 denotes the Recursive Comprehension Axiom (RCA_0), the weakest member of the *big five* subsystems of second order arithmetic (see [10]), and a base system for most of reverse mathematics. In this article, we shall also take RCA_0 as a base system and always assume RCA_0 without explicit reference.

Later, Jockusch [5] proved a series of interesting results concerning complexity of homogeneous sets in terms of arithmetic hierarchy. Moreover, Jockusch constructed a computable 2-coloring of triples, for which every infinite homogeneous set computes the halting problem. In terms of reverse mathematics, Ramsey's theorem for triples (RT_2^3) is equivalent to the Arithmetical Comprehension Axiom (ACA_0), another member of the big five subsystems.

2000 *Mathematics Subject Classification.* 03B30, 03F35, 03D32, 03D80.

The author thanks Chitao Chong and Yue Yang for sharing insights on Ramsey theory and reverse mathematics. He also thanks Hirschfeldt and Slaman for their inspiring lectures on Seetapun's theorem, IMS of NUS and the John Templeton Foundation for organizing and funding his participation in a series of logic programs, and the referees for comments and corrections. This research was partially supported by NSF Grant 11001281 of China and an NCET grant from the Ministry of Education of China.

Jockusch's work left opened whether RT_2^2 implies ACA_0 . This gap was overcome by Seetapun [9], who showed that ACA_0 is strictly stronger than RT_2^2 . In his celebrated proof, Seetapun imposed some complexity conditions on Mathias forcing and got a subset of forcing conditions. In order to prove density lemmas for this subset, he exploited a cone avoidance theorem for Π_1^0 classes by Jockusch and Soare [6], which reflects the power of Π_1^0 classes in controlling complexity. From a computability theoretic point of view, Seetapun's proof shed deep insight into the classical proof of RT_2^2 .

Seetapun's proof was analyzed in Cholak, Jockusch and Slaman [1], where two consequences of RT_2^2 were introduced, namely COH and SRT_2^2 .

Definition 1.1. (1) Let $\vec{R} = (R_n : n \in \omega)$ be a sequence of subsets of ω . An infinite set $X \subseteq \omega$ is \vec{R} -cohesive, if and only if for each n either $X \cap R_n$ or $X - R_n$ is finite.

COH is the assertion that there exists an \vec{R} -cohesive set for every \vec{R} .

(2) A k -coloring $f : [\omega]^2 \rightarrow k$ is *stable* if and only if for every x there exist $i < k$ and y such that $f(x, z) = i$ for all $z > y$.

SRT_k^2 is the assertion that there exists an infinite homogeneous set for every stable k -coloring.

Cholak, Jockusch and Slaman [1] and Mileti [8] showed that RT_2^2 is equivalent to $\text{COH} + \text{SRT}_2^2$. Moreover, Cholak, Jockusch and Slaman obtained a sharp bound on complexity of homogeneous sets for coloring of pairs, and with this sharp bound they built an ω -model of RT_2^2 containing only low_2 sets (X is low_2 if and only if $X'' \equiv_T \emptyset''$). The decomposition of RT_2^2 into COH and SRT_2^2 has become a paradigm of analysis of combinatorial principles relating to RT_2^2 . For example, in a similar vein Hirschfeldt and Shore [4] decomposed the Ascending-Descending-Sequence principle (ADS) into CADS and SADS, and also the Chain-Antichain principle (CAC) into CCAC and SCAC; and respectively, they proved that $\text{CADS} \not\vdash \text{SADS}$, $\text{SADS} \not\vdash \text{CADS}$, $\text{CCAC} \not\vdash \text{SCAC}$ and $\text{SCAC} \not\vdash \text{CCAC}$.

Another interesting aspect of this analysis is that it gives a new proof of RT_2^2 . Actually, computability theoretic techniques are in need for controlling complexity of solutions to certain combinatorial principles, and usually lead to new insight into combinatorial proofs or even new proofs. A recent example is a joint work of Csima and Mileti on Rainbow Ramsey Theorems [2].

Definition 1.2. A function $f : [\omega]^k \rightarrow \omega$ is a b -bounded coloring if $|f^{-1}(c)| \leq b$ for each c . $X \subseteq \omega$ is an f -rainbow if $f \upharpoonright [X]^k$ is injective.

Rainbow Ramsey Theorem. (RRT_b^k) For each b -bounded $f : [\omega]^k \rightarrow \omega$ there exists an infinite f -rainbow.

A well known proof of RRT_b^k is by Galvin. For each b -bounded $f : [\omega]^k \rightarrow \omega$, Galvin defined a dual coloring $g : [\omega]^k \rightarrow b$ such that each g -homogeneous set is an f -rainbow, and then apply RT_b^k (see [2]). Csima and Mileti obtained a new proof of RRT_2^2 . Applying results from algorithmic randomness, Csima and Mileti showed that if R is 2-random relative to Z and $f \leq_T Z$ is 2-bounded then R computes an infinite f -rainbow. From this, they deduced several reverse mathematical consequences, like $\text{RRT}_2^2 \not\vdash \text{RT}_2^2$, $\text{RRT}_2^2 \not\vdash \text{WKL}_0$, etc.

In this paper, we shall present some further investigations of RRT_b^k .

In §3, we present the main result that $\text{RRT}_2^3 \not\vdash \text{ACA}_0$ (Theorem 3.1). As $\text{RT}_2^3 \vdash \text{ACA}_0$ by Jockusch [5], this sounds a little surprising. It then follows from Jockusch's result that RRT_2^3 is strictly weaker than RT_2^3 . As one may expect, Theorem 3.1 is based on some cone avoidance result (Lemma 3.2). By an application of a cone avoidance result for COH, we reduce this cone avoidance for colorings of triples to a strong cone avoidance theorem for 2-bounded colorings of pairs (Theorem 2.1). Theorem 2.1 is strong, in that it gives us rainbows of *low complexity* for colorings of *arbitrary complexity*. We shall see in §2, that the strength of Theorem 2.1 actually comes from some hidden strength of Seetapun's cone-avoidance theorem for RT_2^2 , which was discovered by Dzhafarov and Jockusch [3]. The proof of Theorem 2.1 combines measure theoretic argument from Csimá and Mileti [2] and Mathias forcing in Seetapun's style, and also inductive applications of infinite pigeonhole principle. From a combinatorial viewpoint, the application of COH and inductive applications of pigeonhole principle together amount to inductive applications of RT_2^2 , and such applications of RT_2^2 dispense the need of RT_2^3 in building rainbows for colorings of triples.

In §4, we apply the method in §2 and §3 to obtain some partial cone avoidance result for 2-bounded colorings of quadruples. In §5, we conclude this paper with a conjecture.

Before the presentations of results and proofs, we introduce some notions.

A *model* of second order arithmetic is a pair (M, \mathcal{S}) where M is a first order model of arithmetic and \mathcal{S} is a subset of the powerset of M . An ω -*model* is a model (M, \mathcal{S}) with $M = \omega$. The least ω -model of RCA_0 is (ω, \mathcal{R}) where \mathcal{R} is the set of all computable sets. If $\mathcal{M} = (\omega, \mathcal{S})$ is a model and $X \subseteq \omega$, then $\mathcal{M}[X] = (\omega, \mathcal{S}[X])$, where $\mathcal{S}[X] = \{Z : \exists Y \in \mathcal{S} (Z \leq_T X \oplus Y)\}$. In addition, if $\mathcal{M} \models \text{RCA}_0$ then $\mathcal{M}[X] \models \text{RCA}_0$ for all X .

A *tree* T is a subset of $\omega^{<\omega}$ closed under initial segments. If T is a tree, then $[T]$ denote the set of infinite sequences whose finite initial segments are always in T . A tree T is *X-computably bounded* if there exists an X -computable function h such that $T \cap \omega^n \subseteq D_{h(n)}$ for each n , where D_i is the i -th finite subset of $\omega^{<\omega}$ under some fixed computable coding.

Working with Ramsey-like combinatorial principles, we identify $[X]^{<\omega}$ with the set of strictly increasing sequences in $X^{<\omega}$ and $[X]^\omega$ with the set of infinite strictly increasing sequences from X . We use lower case Greek letters σ, τ, \dots for elements of $[\omega]^{<\omega}$ and $\omega^{<\omega}$, and write $\sigma\tau$ for concatenation of σ and τ . Under the above convention, we may use concatenations for unions of finite sets, e.g., $\sigma\tau = \sigma \cup \tau$ and $\sigma\{x\} = \sigma \cup \{x\}$. We fix a computable bijection $\langle \dots \rangle : [\omega]^{<\omega} \rightarrow \omega$, such that

$$\langle x_0, \dots, x_{n-1} \rangle < \langle y_0, \dots, y_{n-1} \rangle \leftrightarrow \exists i < n (x_i < y_i \wedge \forall j \in (i, n) (x_j = y_j)).$$

For computations using finite oracles, we write $\Phi_e(\sigma; x) \downarrow$ if $\Phi_e(\sigma; x)$ converges in no more than $|\sigma|$ steps. For $Z \subseteq \omega$ and $\sigma \in [\omega]^{<\omega}$, we write $Z \oplus \sigma$ for $(Z \upharpoonright |\sigma|) \oplus \sigma$. So, $\Phi_e(Z \oplus \sigma; x) \downarrow$ means $\Phi_e((Z \upharpoonright |\sigma|) \oplus \sigma; x) \downarrow$, etc.

Below, we summarize some useful results from Seetapun [9] and Cholak, Jockusch and Slaman [1], which will be called the *cone avoidance* of RT_2^2 , COH or SRT_n^2 .

Theorem 1.3. *Suppose that $W \not\leq_T Z$.*

- (1) *If $f : [\omega]^2 \rightarrow 2$ is Z -computable then there exists an infinite f -homogeneous X such that $W \not\leq_T Z \oplus X$.*

- (2) If $\vec{R} = (R_n : n \in \omega)$ is uniformly Z -computable then there exists an infinite \vec{R} -cohesive X such that $W \not\leq_T Z \oplus X$.
- (3) If $g : \omega \rightarrow n$ is Z' -computable then there exist $k < n$ and $X \in [g^{-1}(k)]^\omega$ such that $W \not\leq_T Z \oplus X$.

Dzhafarov and Jockusch [3] observed that a Seetapun-style Mathias forcing could work for Theorem 1.3(3) without the computability condition on g . Alternatively, we can remove this condition by an application of Theorem 1.3(3) itself, in a way similar to [9, §2.4].

Corollary 1.4 (Lemma 5.2(i) in [3]). *Suppose that $W \not\leq_T Z$ and $g : \omega \rightarrow n$ is any finite partition of ω . Then there exist $k < n$ and $X \in [g^{-1}(k)]^\omega$ with $W \not\leq_T Z \oplus X$.*

Proof. Following the proof of Friedberg's Jump Inversion Theorem (see [11, VI.3]), we can build Y such that $W \not\leq_T Y \oplus Z$ and $g \leq_T (Y \oplus Z)'$. Now the desired k and X can be obtained from Theorem 1.3(3). \square

We call Corollary 1.4 as another *Seetapun's cone avoidance* for infinite pigeonhole principle. The strength of the above corollary is the source of the strength of Theorem 2.1. Furthermore, one should note that Corollary 1.4 could be applied to remove the computability condition on \vec{R} in Theorem 1.3(2). As we do not need such a strong result for COH here, we leave this as an exercise for practising Mathias forcing.

We shall need the following theorem by Jockusch and Soare [6].

Theorem 1.5. *If $W \not\leq_T Z$ and $T \leq_T Z$ is an infinite Z -computably bounded tree then there exists $X \in [T]$ with $W \not\leq_T Z \oplus X$.*

We refer readers to Simpson's book [10] for more notions of reverse mathematics and Soare's book [11] for computability theoretic notions.

2. RAINBOWS FOR COLORINGS OF PAIRS

In this section, we prove the following cone avoidance theorem for 2-bounded colorings of pairs. Note that there is no computability theoretic condition on f . In this sense, the following theorem is an analogous of Seetapun's cone avoidance for infinite pigeonhole principle (Corollary 1.4). Actually, the proof needs inductively applications of Corollary 1.4.

Theorem 2.1. *Suppose that $W \not\leq_T Z$. Then every 2-bounded $f : [\omega]^2 \rightarrow \omega$ admits an infinite rainbow H such that $W \not\leq_T H \oplus Z$.*

Below we prove the above theorem. The proof consists of two main steps: firstly we pass from ω to an infinite tail rainbow X such that $W \not\leq_T X \oplus Z$; then we prove the theorem for f with ω being a tail rainbow. A set X is a *tail rainbow* for f , if $f(x_0, x_1) \neq f(y_0, y_1)$ for all $(x_0, x_1), (y_0, y_1) \in [X]^2$ with $x_1 \neq y_1$.

From [2, Proposition 3.3], we learn that the first step is easy for computable f . However, as now we are dealing with arbitrary colorings, we need some measure theoretic argument, which is essentially from [2].

Lemma 2.2. *If $f : [\omega]^2 \rightarrow \omega$ is 2-bounded and R is Martin-Löf random in f then f admits an infinite tail rainbow computable in R .*

In particular, if $W \not\leq_T Z$ then f admits an infinite tail rainbow X with $W \not\leq_T X \oplus Z$.

Proof. Let $h(k) = k$ for $k \leq 2$. For $k > 2$, let

$$h(k) = h(k-1) + \min\{2^m : 2^m \geq 2^{k-1} \frac{(k-1)(k-2)}{2}\}.$$

Let S be the set of $\sigma \in [\omega]^{<\omega}$ such that $h(k) \leq \sigma(k) < h(k+1)$ for all $k < |\sigma|$, and let

$$T = \{\sigma \in S : \sigma \text{ is a tail rainbow for } f\}.$$

Given any $\sigma \in [\omega]^{<\omega}$, we have

$$|f([\sigma]^2)| \leq |[\sigma]^2| = \frac{|\sigma|(|\sigma|-1)}{2}.$$

By 2-boundedness of f ,

$$|\{x > \max \sigma : \exists w (w < x \wedge f(w, x) \in f([\sigma]^2))\}| \leq |f([\sigma]^2)| \leq \frac{|\sigma|(|\sigma|-1)}{2}.$$

Thus, if $\sigma \in T$ then

$$|\{x : \sigma\{x\} \in T\}| \geq (1 - 2^{-|\sigma|})|\{x : \sigma\{x\} \in S\}|.$$

It follows that, for all k ,

$$|T \cap [\omega]^k| \geq 2^{-1} |S \cap [\omega]^k|.$$

By the definition of S , we can computably map $[S]$ onto 2^ω by computably mapping S to $2^{<\omega}$ as following: if $\sigma \in S$ is mapped to $\nu \in 2^{<\omega}$, then $\sigma\{x\} \in S$ is mapped to $\nu\xi$, so that x is the $(\sum_{\xi(i)=1} 2^i)$ -th number $\geq h(|\sigma|)$. Under such mapping, $[T]$ is mapped onto a Π_1^f subclass of 2^ω with positive measure. By the corollary of Lemma 3 in Kučera [7], every R which is Martin-Löf random in f , computes some $X \in [T]$ which is a tail f -rainbow.

On the other hand, if $W \not\leq_T Z$ then $\{Y \in 2^\omega : W \leq_T Y \oplus Z\}$ is null. So we can pick R and $X \in [T]$ such that R is Martin-Löf random in f , $X \leq_T R$ and $W \not\leq_T R \oplus Z$. \square

With Lemma 2.2, we proceed to the second step.

Lemma 2.3. *Suppose that $W \not\leq_T Z$. If f is a 2-bounded coloring of pairs with ω being a tail rainbow, then there exists an infinite f -rainbow G such that $W \not\leq_T G \oplus Z$.*

Below, we fix a coloring f as in the above lemma and build a desired rainbow by Mathias forcing. The plan is as following:

- (1) As f is of arbitrary complexity, we can not directly consult f in the construction. So, we define a Π_1^0 class (say \mathcal{A}) which captures f in some sense.
- (2) Instead of asking questions about f , we ask questions like: whether \mathcal{A} contains some element satisfying certain Π_1^Z property (say φ). Roughly, $\varphi(g)$ holds for a coloring $g \in \mathcal{A}$, if in a measure theoretic sense most g -rainbows do not contain splitting computations.
- (3) If \mathcal{A} does contain such an element, then we can pick a cone-avoiding $g \in \mathcal{A}$ by Jockusch-Soare's Theorem 1.5. By a measure theoretic argument like [2, §3], we can obtain a cone-avoiding g -rainbow which contains no splitting computations. With this rainbow, we can extend a given Mathias condition to one which forces some Π_1^Z statement (e.g., $\Phi_e(Z \oplus G; x) \uparrow$) and has desirable complexity.

- (4) Otherwise, f is a particular element of \mathcal{A} which satisfies a Σ_1^Z property $(\neg\varphi)$. From this fact, we can extend a given Mathias condition in a finitary way to force a Σ_1^Z statement, like $\Phi_e(Z \oplus G; x) \downarrow \neq W(x)$.

People who are familiar with [9] can find that the above plan is a variation of Seetapun's argument.

We begin with the definition of \mathcal{A} .

Definition 2.4. Let $g : [\omega]^2 \rightarrow \omega$ be a 2-bounded coloring. If

$$\forall (x_0, x_1) \in [\omega]^2 (g(x_0, x_1) = \langle \tilde{g}(x_0, x_1), x_1 \rangle)$$

where $\tilde{g}(x_0, x_1) = \min\{x \leq x_0 : g(x, x_1) = g(x_0, x_1)\}$, then g is *normal*.

Let \mathcal{A} be the set of all normal 2-bounded colorings of $[\omega]^2$.

Note that in this paper the notion of normal colorings is different from Csima and Mileti [2]. Let us briefly justify our terminology. If g is a 2-bounded coloring of pairs with ω being a tail rainbow, then we can define \tilde{g} from g as in the above definition and let

$$\bar{g}(x_0, x_1) = \langle \tilde{g}(x_0, x_1), x_1 \rangle.$$

If $\bar{g}(x, x_1) = \bar{g}(y, x_1) = \langle w, x_1 \rangle$ and $x < y$, then $w = \tilde{g}(x, x_1) = \tilde{g}(y, x_1) \leq x$ and thus $g(w, x_1) = g(x, x_1) = g(y, x_1)$. By 2-boundedness of g , $w = x$ and thus we have 2-boundedness of \bar{g} and $\bar{g} \in \mathcal{A}$. It also follows that g - and \bar{g} -rainbows coincide. On the other hand, suppose that $h_0, h_1 \in \mathcal{A}$ and $h_0(x, y) < h_1(x, y)$. Then $h_0(x, y) < h_1(x, y) \leq \langle x, y \rangle$ and $h_0(x, y) = \langle u, y \rangle$ for some $u < x$. As h_0 and h_1 are normal, $\{u, x, y\}$ is a rainbow for h_1 but not for h_0 . So, every coloring like f is equivalent to a unique element of \mathcal{A} , in the sense that they have same rainbows.

Hence, we can safely assume that $f \in \mathcal{A}$. Using some effective coding, \mathcal{A} can be identified with a Π_1^0 subset of Cantor space.

We proceed to define a suitable subset of Mathias conditions. Let us begin from recalling standard Mathias forcing.

Definition 2.5. A *Mathias condition* is a pair $(\sigma, X) \in [\omega]^{<\omega} \times [\omega]^\omega$ such that $\max \sigma < \min X$. For each Mathias condition (σ, X) , let

$$B(\sigma, X) = \{S \in [\omega]^\omega : \sigma \subseteq S \subseteq \sigma \cup X\}.$$

Given two Mathias conditions (σ, X) and (τ, Y) , $(\tau, Y) \leq_M (\sigma, X)$ if and only if $B(\tau, Y) \subseteq B(\sigma, X)$.

Let (σ, X) be a Mathias condition. We write $(\sigma, X) \Vdash \Phi_e(\dot{Z} \oplus \dot{G}) \neq \dot{W}$, if and only if $\Phi_e(Z \oplus G) \neq W$ for every $G \in B(\sigma, X)$.

We need to find a descending sequence of Mathias condition $((\sigma_n, X_n) : n \in \omega)$ such that $G = \bigcup_n \sigma_n$ is an infinite f -rainbow and $(\sigma_n, X_n) \Vdash \Phi_n(\dot{Z} \oplus \dot{G}) \neq \dot{W}$. For forcing incomputability requirements, it is natural to require that $W \not\leq_T Z \oplus X_n$. For rainbow requirement, we need some further restriction on the infinite tails of Mathias conditions.

For $\sigma \in [\omega]^{<\omega}$ and $g \in \mathcal{A}$, let the set below collect all *viable numbers*:

$$V(\sigma, g) = \{x > \max \sigma : \sigma \cup \{x\} \text{ is a rainbow for } g\}.$$

For a Mathias condition (σ, X) , let

$$\mathcal{A}_{\sigma, X} = \{g \in \mathcal{A} : X \subseteq V(\sigma, g)\} \in \Pi_1^X.$$

To satisfy rainbow requirement, we should work with (σ, X) having $f \in \mathcal{A}_{\sigma, X}$. Colorings in $\mathcal{A}_{\sigma, X}$ are considered as *acceptable*.

Definition 2.6. A Mathias condition (σ, X) is *admissible*, if and only if $f \in \mathcal{A}_{\sigma, X}$ and $W \not\leq_T Z \oplus X$.

If (3) in the above plan holds for some $g \in \mathcal{A}_{\sigma, X}$, we need to pass from X to a g -rainbow with desirable complexity. To this end, we define a tree of g -rainbows.

Let $b_0 = 1$. For each $l > 0$, let $b_l = \min\{2^b : 2^b \geq 2^{l+3}(|\sigma| + l)\}$.

Definition 2.7. Let (σ, X) be a Mathias condition. If $g \in \mathcal{A}_{\sigma, X}$, then we associate to (σ, X, g) a tree, denoted by $T(\sigma, X, g) \subseteq [\omega]^{<\omega}$, which is defined by induction as below:

- (1) $\emptyset \in T(\sigma, X, g)$.
- (2) Suppose that $\tau \in T(\sigma, X, g) \cap [\omega]^l$. If x is among the least b_l elements in $V(\sigma\tau, g) \cap X$ then $\tau\{x\} \in T(\sigma, X, g)$.

Some facts about $T(\sigma, X, g)$ follow easily from the definition:

- (T1) $T(\sigma, X, g)$ is $(g \oplus X)$ -computable uniformly in (σ, X, g) .
- (T2) $[T(\sigma, X, g)]$ is a compact subset of Baire Space ω^ω .
- (T3) $|T(\sigma, X, g) \cap [\omega]^l| \leq \bar{b}_l = \prod_{i < l} b_i$ for each $l > 0$.
- (T4) $\sigma\tau$ is a g -rainbow for each $\tau \in T(\sigma, X, g)$.

$T(\sigma, X, g)$ is similar to T_f in [2, §3]. But $T(\sigma, X, g)$ is compact, while T_f in [2, §3] is very wild. The compactness is needed in Lemma 2.11 below. We may prove that $T(\sigma, X, g)$ is bushy in a way similar to [2, §3]. However, we shall need some different calculation.

Lemma 2.8. *Suppose that (σ, X) is a Mathias condition and $g \in \mathcal{A}_{\sigma, X}$. For each l , let $y \in X$ be such that $y > \max\{\max \tau : \tau \in T(\sigma, X, g) \cap [\omega]^l\}$. Then*

$$|\{\tau \in T(\sigma, X, g) \cap [\omega]^l : y \notin V(\sigma\tau, g)\}| < \frac{\bar{b}_l}{4}.$$

Proof. Let $T = T(\sigma, X, g)$. For non-empty $\tau \in [X]^{<\omega}$, let $x_\tau = \max \tau$ and $\tau^- = \tau - \{x_\tau\}$. For each l and y , let

$$N_{l,y} = \{\tau \in T \cap [\omega]^l : y \notin V(\sigma\tau, g)\}.$$

We show by induction on l that if $y \in X$ and $y > \max\{x_\tau : \tau \in T \cap [\omega]^l\}$ then

$$|N_{l,y}| < \left(\frac{1}{4} - \frac{1}{2^{l+2}}\right)\bar{b}_l.$$

Let $y \in X$ be such that $y > \max\{x_\tau : \tau \in T \cap [\omega]^{l+1}\}$, and let

$$N_{l+1,y,0} = \{\tau \in T \cap [\omega]^{l+1} : \tau^- \in N_{l,y}\}.$$

Then $|N_{l+1,y,0}| \leq |N_{l,y}|b_l$, by the definition of T . Let

$$N_{l+1,y,1} = N_{l+1,y} - N_{l+1,y,0}.$$

Suppose that $\rho \in T \cap [\omega]^l - N_{l,y}$ and $\rho\{x\} \in N_{l+1,y,1}$. Then $y \in V(\sigma\rho, g) - V(\sigma\rho\{x\}, g)$. As both $\sigma\rho\{x\}$ and $\sigma\rho\{y\}$ are g -rainbows, $g(x, y) = g(u, v)$ for some $(u, v) \in [\sigma\rho\{x, y\}]^2 - \{(x, y)\}$. As ω is a tail g -rainbow, $v = y$. So, $g(x, y) = g(u, y)$ for some $u \in \sigma\rho$. As g is 2-bounded, there are at most $|\sigma| + l$ many x 's such that $\rho\{x\} \in N_{l+1,y,1}$. So

$$|N_{l+1,y,1}| \leq (|T \cap [\omega]^l| - |N_{l,y}|)(|\sigma| + l) \leq (\bar{b}_l - |N_{l,y}|)(|\sigma| + l).$$

Hence,

$$\begin{aligned}
|N_{l+1,y}| &\leq |N_{l,y}|b_l + (\bar{b}_l - |N_{l,y}|)(|\sigma| + l) \\
&\leq |N_{l,y}|b_l + \frac{1}{2^{l+3}}b_l(\bar{b}_l - |N_{l,y}|) \\
&< \frac{\bar{b}_{l+1}}{2^{l+3}} + (1 - \frac{1}{2^{l+3}})(\frac{1}{4} - \frac{1}{2^{l+2}})\bar{b}_{l+1} \\
&< (\frac{1}{4} - \frac{1}{2^{l+3}})\bar{b}_{l+1}.
\end{aligned}$$

This proves the lemma. \square

Lemma 2.9. *Suppose that (σ, X) is a Mathias condition and $g \in \mathcal{A}_{\sigma, X}$. Then*

$$|T(\sigma, X, g) \cap [\omega]^l| > \frac{3}{4}\bar{b}_l \text{ for all } l.$$

Proof. For $l > 0$, let

$$M_l = \{\tau \in T(\sigma, X, g) \cap [\omega]^l : V(\sigma\tau, g) \text{ is infinite}\}.$$

By the above lemma,

$$l > 0 \rightarrow |M_l| > \frac{3}{4}\bar{b}_l.$$

This proves the lemma. \square

By the above lemma and the definition of $T(\sigma, X, g)$, if $g \in \mathcal{A}_{\sigma, X}$ then we can $X \oplus g$ -computably map $[T(\sigma, X, g)]$ onto some $[S] \subseteq 2^\omega$ with measure at least three quarters, where S is a binary tree computably enumerable in $X \oplus g$. The mapping goes as following: if $\tau \in T(\sigma, X, g)$ is mapped to $\nu \in S$, then $\tau\{x\} \in T(\sigma, X, g)$ is mapped to $\nu\xi \in S$, so that x is the $(\sum_{\xi(i)=1} 2^i)$ -th element of $V(\sigma\tau, g) \cap X$. As $V(\sigma\tau, g) \cap X$ could be empty, the resulting S can only be $\Sigma_1^{X \oplus g}$, although $T(\sigma, X, g)$ is $X \oplus g$ -computable.

We are ready to extend an admissible condition to another that forces an incomputability requirement. To this end, we use splitting computations as usual.

Definition 2.10. A finite sequence ρ (e, Z) -splits over σ , if and only if $\rho = \sigma\tau$ for some τ and there are different τ_0 and τ_1 such that both τ_0 and τ_1 are subsets of τ and $\Phi_e(Z \oplus (\sigma\tau_0); x) \downarrow \neq \Phi_e(Z \oplus (\sigma\tau_1); x) \downarrow$ for some x .

If $g \in \mathcal{A}_{\sigma, X}$ is such that

$$|\{\tau \in T(\sigma, X, g) \cap [\omega]^l : \sigma\tau \text{ } (e, Z)\text{-splits over } \sigma\}| < \frac{\bar{b}_l}{2}$$

for all l , then g -rainbows (e, Z) -split over (σ, X) with low probability.

It is a $\Pi_1^{Z \oplus X \oplus g}$ question whether g -rainbows (e, Z) -split over (σ, X) with low probability. This complexity bound is due to $T(\sigma, X, g)$ being nicely bounded. This explains why we define $T(\sigma, X, g)$ in a way different from T_f in [2].

Lemma 2.11. *Suppose that (σ, X) is an admissible Mathias condition. For each e , there exists an admissible $(\tau, Y) \leq_M (\sigma, X)$ such that $(\tau, Y) \Vdash \Phi_e(\dot{Z} \oplus \dot{G}) \neq \dot{W}$.*

Proof. Let \mathcal{U} be the set of $g \in \mathcal{A}_{\sigma, X}$ such that g -rainbows (e, Z) -split over (σ, X) with low probability. Clearly, \mathcal{U} is $\Pi_1^{Z \oplus X}$. There are two cases.

Case 1: $\mathcal{U} \neq \emptyset$. By Theorem 1.5 of Jockusch and Soare, there exists $g \in \mathcal{U}$ such that $W \not\leq_T Z \oplus g \oplus X$. Let

$$T = T(\sigma, X, g) \cap \{\tau \in [\omega]^{<\omega} : \sigma\tau \text{ does not } (e, Z)\text{-split over } \sigma\}.$$

As $T(\sigma, X, g)$ is computable in $g \oplus X$, T is computable in $Z \oplus X \oplus g$. By Lemma 2.9 and that g -rainbows (e, Z) -split over (σ, X) with low probability,

$$\forall l (|T \cap [\omega]^l| \geq \frac{\bar{b}_l}{4}).$$

Combining the above inequality and the remark following Lemma 2.9, and by an application of the corollary of Lemma 3 in Kučera [7], if R is 2-random in $Z \oplus g \oplus X$ then $Z \oplus g \oplus X \oplus R$ computes some $Y \in [T]$. As $W \not\leq_T Z \oplus g \oplus X$, there exist Y and R such that R is 2-random in $Z \oplus g \oplus X$, $Y \leq_T Z \oplus g \oplus X \oplus R$, $Y \in [T]$ and $W \not\leq_T Z \oplus Y$. Note that each path of T is a subset of X . So, (σ, Y) is an admissible extension of (σ, X) .

It remains to show that $(\sigma, Y) \Vdash \Phi_e(\dot{Z} \oplus \dot{G}) \neq \dot{W}$. Suppose that for each x there exists $\tau \in [Y]^{<\omega}$ such that $\Phi_e(Z \oplus (\sigma\tau); x) \downarrow$. Then there must be some x such that $\Phi_e(Z \oplus (\sigma\tau); x) \neq W(x)$ whenever $\Phi_e(Z \oplus (\sigma\tau); x) \downarrow$ for $\tau \in [Y]^{<\omega}$, as $W \not\leq_T Z \oplus Y$ and $\sigma \cup Y$ does not (e, Z) -split over σ .

Case 2: $\mathcal{U} = \emptyset$. In particular $f \notin \mathcal{U}$. By the definition of \mathcal{U} , there exist some $l > 0$ and

$$\{\tau_i : i < b = \frac{\bar{b}_l}{2}\} \subseteq \{\tau \in T(\sigma, X, f) \cap [\omega]^l : \sigma\tau \text{ } (e, Z)\text{-splits over } \sigma\}.$$

Let $m = \max \bigcup_{i < b} \tau_i$. By Lemma 2.8, for each $y \in X - [0, m]$ there exists $i < b$ such that $y \in V(\sigma\tau_i, f)$. Let $p(y)$ be the least such i . So, p is a finite partition of $X - [0, m]$. By Seetapun's cone avoidance for infinite pigeonhole principle (Corollary 1.4), there exist $i < b$ and $Y \in [X \cap p^{-1}(i)]^\omega$ such that $W \not\leq_T Z \oplus X \oplus Y$. It follows that $f \in \mathcal{A}_{\sigma\tau_i, Y}$. As $\sigma\tau_i$ (e, Z) -splits over σ , there are x , π_0 and π_1 such that $\sigma \subseteq \pi_j \subseteq \sigma\tau_i$ for $j < 2$ and $\Phi_e(Z \oplus \pi_0; x) \downarrow \neq \Phi_e(Z \oplus \pi_1; x) \downarrow$. Take $\tau = \pi_j$ such that $\Phi_e(Z \oplus \pi_j; x) \neq W(x)$.

So, $(\tau, Y) \leq_M (\sigma, X)$ is admissible and $(\tau, Y) \Vdash \Phi_e(\dot{Z} \oplus \dot{G}) \neq \dot{W}$. \square

An argument similar to the second case in the above proof allows us to extend the finite heads of admissible Mathias conditions.

Lemma 2.12. *If (σ, X) is an admissible Mathias condition then there exists an admissible $(\tau, Y) \leq_M (\sigma, X)$ such that $\sigma \subset \tau$.*

Proof. Let $l = |\sigma|$ and x_0, \dots, x_l be the first $l+1$ elements of X in ascending order. For each $y \in X \cap (x_l, \infty)$, let $p(y)$ be the least $i \leq l$ such that $y \in V(\sigma\{x_i\}, f)$. By the 2-boundedness of f , p is total on $X \cap (x_l, \infty)$. By Seetapun's cone avoidance for infinite pigeonhole principle again, there exist $i \leq l$ and $Y \in [X \cap p^{-1}(i)]^\omega$ such that $W \not\leq_T Z \oplus X \oplus Y$. Clearly, $(\sigma\{x_i\}, Y)$ is as desired. \square

By Lemmata 2.11 and 2.12, we can obtain a descending sequence of Mathias conditions $((\sigma_n, X_n) : n \in \omega)$ such that $(\sigma_n, X_n) \Vdash \Phi_n(\dot{Z} \oplus \dot{G}) \neq \dot{W}$ and $\sigma_n \subset \sigma_{n+1}$ for all n . So, $G = \bigcup_n \sigma_n$ is a desired f -rainbow. This proves Lemma 2.3.

Theorem 2.1 follows immediately.

3. RAINBOWS FOR COLORINGS OF TRIPLES

In this section, we prove the main theorem.

Theorem 3.1. $\text{RRT}_2^3 \not\models \text{ACA}_0$. Thus RRT_2^3 is strictly weaker than RT_2^3 .

To this end, for each non-computable W we build an ω -model $\mathcal{M} = (\omega, \mathcal{S})$ such that $\mathcal{M} \models \text{RCA}_0 + \text{RRT}_2^3$ and $W \notin \mathcal{S}$. In particular, taking W to be a non-computable arithmetic set, yields that $\mathcal{M} \not\models \text{ACA}_0$.

The key is to find an f -rainbow X with $W \not\leq_T Z \oplus X$, whenever $W \not\leq_T Z$ and $f : [\omega]^3 \rightarrow \omega$ is 2-bounded and Z -computable. If we achieve this, then we can inductively build a desired \mathcal{S} .

We state the above key step as a lemma.

Lemma 3.2. *If $W \not\leq_T Z$ and Z computes a 2-bounded $f : [\omega]^3 \rightarrow \omega$, then there exists an infinite f -rainbow X such that $W \not\leq_T Z \oplus X$.*

Proof. Let W, Z and f be as above. By [2, Proposition 3.3], we may assume that ω is a 1-tail f -rainbow, i.e., $f(u, v, w) \neq f(x, y, z)$ whenever $w \neq z$. Let

$$I = \{(u, v, x, y) \in [\omega]^2 \times [\omega]^2 : \langle u, v \rangle < \langle x, y \rangle\}.$$

Let $\bar{f}(x, y, z) = \min\{\langle u, v \rangle : f(u, v, z) = f(x, y, z)\}$ for each $(x, y, z) \in [\omega]^3$, and let

$$R_{u,v,x,y} = \{s > y : \bar{f}(x, y, s) = \langle u, v \rangle\}$$

for $(u, v, x, y) \in I$. Note that $\bar{f}(x, y, z) \leq \langle x, y \rangle$. By the cone avoidance of COH, there exists C such that C is cohesive for $\vec{R} = (R_{u,v,x,y} : (u, v, x, y) \in I)$ and $W \not\leq_T Z \oplus C$.

For $(x, y) \in [C]^2$, let

$$\hat{f}(x, y) = \begin{cases} \langle u, v \rangle, & \langle u, v \rangle = \lim_{s \in C} \bar{f}(x, y, s) \wedge (u, v) \in [C]^2; \\ \langle x, y \rangle, & \text{otherwise.} \end{cases}$$

By cohesiveness of C , \hat{f} is well defined. Moreover, if $\hat{f}(x, y) = \langle u, v \rangle$ and $(u, v) \in [C]^2$, then $\langle u, v \rangle \leq \langle x, y \rangle$ and $\bar{f}(x, y, s) = \langle u, v \rangle$ for sufficiently large $s \in C$. Thus $f(x, y, s) = f(u, v, s)$ for sufficiently large $s \in C$. So, if $\hat{f}(x, y) = \hat{f}(x', y')$ then $f(x, y, s) = f(x', y', s)$ for sufficiently large $s \in C$. As f is 2-bounded, \hat{f} is also 2-bounded. By Theorem 2.1, there exists an infinite \hat{f} -rainbow G such that $G \subseteq C$ and $W \not\leq_T Z \oplus G$.

We define a desired X as a subset of G by induction. Let $X_0 = \emptyset$. Suppose that $X_n \in [G]^{<\omega}$ is defined and X_n is a rainbow for f .

Claim 3.3. *For all sufficiently large $a \in G$, $X_n \cup \{a\}$ is a rainbow for f .*

Proof of Claim 3.3. As X_n is finite and G is \vec{R} -cohesive, there exists s_0 such that $\bar{f}(x, y, a) = \lim_{s \in C} \bar{f}(x, y, s)$ for all $a \in G \cap (s_0, \infty)$ and $(x, y) \in [X_n]^2$.

For $a \in G \cap (s_0, \infty)$, if there are (x, y) and (x', y') in $[X_n]^2$ such that $\langle x, y \rangle < \langle x', y' \rangle$ and $f(x, y, a) = f(x', y', a)$, then $\bar{f}(x', y', a) = \langle x, y \rangle$ and $\hat{f}(x', y') = \langle x, y \rangle = \hat{f}(x, y)$. But this is impossible, as $X_n \subset G$ is a rainbow for \hat{f} .

So, $f(x, y, a) \neq f(x', y', a)$ for distinct (x, y) and (x', y') in $[X_n]^2$ and all $a \in G \cap (s_0, \infty)$. As X_n is an f -rainbow and ω is a 1-tail f -rainbow, $X_n \cup \{a\}$ is an f -rainbow. \square

By the above claim, let a_n be the least $a \in G \cap (\max X_n, \infty)$ such that $X_n \cup \{a\}$ is a rainbow for f . Let $X_{n+1} = X_n \cup \{a_n\}$.

So, $X = \bigcup_n X_n$ is an infinite f -rainbow. As $X \leq_T Z \oplus G$, $W \not\leq_T Z \oplus X$. \square

Proof of Theorem 3.1. Fix a non-computable W . By inductive applications of Lemma 3.2, we can build a sequence $(\mathcal{M}_n : n < \omega)$ such that

- (1) \mathcal{M}_0 is the least ω -model of RCA_0 ;
- (2) Each $\mathcal{M}_{n+1} = \mathcal{M}_n[X_n]$ for some X_n ;
- (3) For all $n > 0$, $W \not\leq_T \bigoplus_{i < n} X_i$;
- (4) If $f : [\omega]^3 \rightarrow \omega$ is a 2-bounded coloring in \mathcal{M}_n then there exists some $m \geq n$ such that X_m is an infinite f -rainbow;

Then $\mathcal{M} = \bigcup_n \mathcal{M}_n$ is a model of $\text{RCA}_0 + \text{RRT}_2^3$ and W is not in \mathcal{M} .

In particular, taking W to be the halting problem, yields that $\mathcal{M} \not\models \text{ACA}_0$. \square

Combining the above proof with the proof of Seetapun's Theorem [9, Theorem 2.1], we get the following corollary.

Corollary 3.4. $\text{WKL}_0 + \text{RT}_2^2 + \text{RRT}_2^3 \not\models \text{ACA}_0$.

4. TAIL RAINBOWS FOR COLORINGS OF QUADRUPLES

In this section, we present some partial answer to the question whether $\text{RRT}_2^n \vdash \text{ACA}_0$ for some $n > 3$. We obtain a cone avoidance result that every computable 2-bounded coloring of $[\omega]^4$ admits a cone-avoiding infinite rainbow-like set.

If $k \leq n$, X and $f : [\omega]^{n+1} \rightarrow \omega$ are such that $f(x_0, x_1, \dots, x_n) \neq f(y_0, y_1, \dots, y_n)$ for (x_0, x_1, \dots, x_n) and (y_0, y_1, \dots, y_n) in $[X]^{n+1}$ with distinct (x_{n-k+1}, \dots, x_n) and (y_{n-k+1}, \dots, y_n) , then we say that X is a k -tail f -rainbow. A *tail f -rainbow* is just a k -tail f -rainbow for $k = n$.

Theorem 4.1. *Suppose that $W \not\leq_T Z$ and $f : [\omega]^4 \rightarrow \omega$ is a 2-bounded coloring computable in Z . Then there exists an infinite tail f -rainbow X such that $W \not\leq_T X \oplus Z$.*

We follow the proof of Lemma 3.2. By [2, Proposition 3.3] and the cone avoidance of COH, it suffices to prove the following cone avoidance result, which is an analogous of Theorem 2.1.

Lemma 4.2. *Suppose that $W \not\leq_T Z$ and $\hat{f} : [\omega]^3 \rightarrow \omega$ is 2-bounded. Then there exists an infinite tail \hat{f} -rainbow X such that $W \not\leq_T X \oplus Z$.*

Fix W, Z and \hat{f} as in the above lemma. By an argument similar to that of Lemma 2.2, we can assume that ω is a 1-tail rainbow for \hat{f} .

To apply Mathias forcing as in §2, we define a class of colorings. For a 2-bounded $g : [\omega]^3 \rightarrow \omega$, if

$$\forall (x_0, x_1, x_2) \in [\omega]^3 (g(x_0, x_1, x_2) = \langle \tilde{g}(x_0, x_1, x_2), x_2 \rangle)$$

where $\tilde{g}(x_0, x_1, x_2) = \min\{\langle y_0, y_1 \rangle : g(y_0, y_1, x_2) = g(x_0, x_1, x_2)\}$, then g is *semi-normal*. Let \mathcal{B} be the set of all semi-normal 2-bounded coloring of $[\omega]^3$. Clearly, ω is a 1-tail rainbow for every $g \in \mathcal{B}$ and \mathcal{B} can be identified as a Π_1^0 subset of Cantor space under some effective coding. Moreover, every 2-bounded coloring of triples with ω being a 1-tail rainbow is equivalent to some element of \mathcal{B} , in the sense of having same tail rainbows. So, we may assume that $\hat{f} \in \mathcal{B}$.

Fix $\sigma \in [\omega]^{<\omega}$ and $g \in \mathcal{B}$. Let $tV(\sigma, g)$ be the set of *tail viable* numbers, i.e., $x \in tV(\sigma, g)$ if and only if $\sigma\{x\}$ is a tail g -rainbow. For a Mathias condition (σ, X) , let

$$\mathcal{B}_{\sigma, X} = \{g \in \mathcal{B} : X \subseteq tV(\sigma, g)\} \in \Pi_1^X.$$

If $g \in \mathcal{B}_{\sigma, X}$ then we associate a tree $\tilde{T}(\sigma, X, g) \subseteq [X]^{<\omega}$ to (σ, X, g) . $\tilde{T}(\sigma, X, g)$ is defined by induction as following.

- (1) $\emptyset \in \tilde{T}(\sigma, X, g)$,
- (2) If $\tau \in \tilde{T}(\sigma, X, g)$ and x is among the first $c_{|\sigma\tau|}$ many elements in $tV(\sigma\tau, g) \cap X \cap (\max \tau, \infty)$ where $c_k = \min\{2^c : 2^c \geq 2^{k+3} \binom{k}{2}\}$, then $\tau\{x\} \in \tilde{T}(\sigma, X, g)$.

$\tilde{T}(\sigma, X, g)$ is similar to the trees in §2. If $\tau \in \tilde{T}(\sigma, X, g)$ then $\sigma\tau$ is a tail \hat{f} -rainbow.

Fix $\tau \in \tilde{T}(\sigma, X, g)$ and $y \in X \cap tV(\sigma\tau, g)$. If $\tau\{x\} \in \tilde{T}(\sigma, X, g)$ and $y \in tV(\sigma\tau, g) - tV(\sigma\tau\{x\}, g)$, then there are $w, u, v \in \sigma\tau$ such that $g(w, x, y) = g(u, v, y)$. By 2-boundedness of g , for our fixed y and τ , there could be at most $\binom{|\sigma\tau|}{2}$ many x 's as above. This simple calculation allows us to establish something like Lemmata 2.8 and 2.9 for $\tilde{T}(\sigma, X, g)$. So, with this new family of trees, we can construct an appropriate Mathias generic as in §2 and thus obtain a tail \hat{f} -rainbow G such that $W \not\leq_T Z \oplus G$. The details are left to the reader.

Actually, we can slightly generalize Theorem 4.1 to the following corollary.

Corollary 4.3. *Suppose that $W \not\leq_T Z$ and $f : [\omega]^{n+4} \rightarrow \omega$ is 2-bounded and Z -computable. Then there exists an infinite 3-tail f -rainbow X such that $W \not\leq_T X \oplus Z$.*

5. A CONJECTURE

Inspired by [1, Theorem 3.1], we may wonder whether we can control triple jumps of rainbows for colorings of triples. This is recently confirmed by the author in an upcoming work: every computable 2-bounded coloring of triples admits a low₃ infinite rainbow X (i.e., $X''' \equiv_T \emptyset'''$). As a consequence of this answer and the lower complexity bound in [2, Theorem 2.5], $\text{RRT}_2^3 \not\vdash \text{RRT}_2^4$. However, the proof for controlling triple jumps is complicated, and apparently hard to be adapted to override cone avoidance results here.

Finally I boldly conjecture the following.

Conjecture 5.1. *For all n , $\text{RCA}_0 + \text{RRT}_2^n \not\vdash \text{ACA}_0$.*

REFERENCES

- [1] Peter A. Cholak, Carl G. Jockusch, and Theodore A. Slaman. On the strength of Ramsey's theorem for pairs. *J. Symbolic Logic*, 66(1):1–55, 2001.
- [2] Barbara Csima and Joseph Mileti. The strength of the rainbow Ramsey theorem. *Journal of Symbolic Logic*, 74(4):1310–1324, 2009.
- [3] Damir D. Dzhalalov and Carl G. Jockusch, Jr. Ramsey's theorem and cone avoidance. *J. Symbolic Logic*, 74(2):557–578, 2009.
- [4] Denis R. Hirschfeldt and Richard A. Shore. Combinatorial principles weaker than Ramsey's theorem for pairs. *J. Symbolic Logic*, 72(1):171–206, 2007.
- [5] Carl G. Jockusch, Jr. Ramsey's theorem and recursion theory. *J. Symbolic Logic*, 37(2):268–280, 1972.
- [6] Carl G. Jockusch, Jr. and Robert I. Soare. Π_1^0 classes and degrees of theories. *Trans. Amer. Math. Soc.*, 173:33–56, 1972.
- [7] Antonín Kučera. Measure, Π_1^0 -classes and complete extensions of PA. In *Recursion theory week (Oberwolfach, 1984)*, volume 1141 of *Lecture Notes in Math.*, pages 245–259. Springer, Berlin, 1985.

- [8] Joseph Roy Mileti. *Partition theorems and computability theory*. ProQuest LLC, Ann Arbor, MI, 2004. Thesis (Ph.D.)—University of Illinois at Urbana-Champaign.
- [9] David Seetapun and Theodore A. Slaman. On the strength of Ramsey’s theorem. *Notre Dame J. Formal Logic*, 36(4):570–582, 1995. Special Issue: Models of arithmetic.
- [10] Stephen G. Simpson. *Subsystems of Second Order Arithmetic*. Perspectives in Mathematical Logic. Springer-Verlag, Berlin, 1999.
- [11] Robert I. Soare. *Recursively Enumerable Sets and Degrees*. Perspectives in Mathematical Logic. Springer-Verlag, Heidelberg, 1987.
- [12] E. Specker. Ramsey’s Theorem does not hold in recursive set theory. In *Logic Colloquium; 1969 Manchester*, pages 439–442, 1971.

INSTITUTE OF LOGIC AND COGNITION AND DEPARTMENT OF PHILOSOPHY, SUN YAT-SEN UNIVERSITY, 135 XINGANG XI ROAD, GUANGZHOU 510275, P.R. CHINA
E-mail address: `wwang.cn@gmail.com`